Computing with finite semigroups: part II

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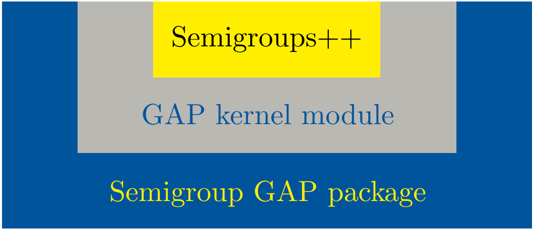
The Semigroups package (version 2.6) - overview

- methods for semigroups, in particular semigroups of transformation, partial permutations, partitions, matrices over finite fields and Rees 0-matrix semigroups
- is (often) faster and (always) more versatile than other software for computing with semigroups
- can calculate Green's classes, size, elements, group of units, minimal ideal, and test membership, find inverses of a regular elements
- factorizing elements over the generators
- testing if a semigroup satisfies a particular property, such as if it is regular, simple, inverse, completely regular
- functions to read/write large collections of element from/to a file
- change representations of semigroups
- compute small generating sets, the maximal subsemigroups of a semigroup, the character table of a inverse semigroup, a smaller degree partial perm representation of an inverse semigroup, the normalizer of a semigroup in a permutation group, produce pictures of the Green's structure of a semigroup, the free inverse semigroup, graph inverse semigroups, free bands, congruences, . . .

The Semigroups package (version 3.0) - overview



- Semigroups++: a standalone C/C++ library containing (so far!) an implementation of the Froidure-Pin Algorithm (including a multithreaded version)

- GAP kernel module: C kernel module for the Semigroups package interface with Semigroups++.

- Semigroups: the GAP package with all of the features from the previous slide for all semigroups in GAP.

The Semigroups package (version 3.0) - overview

- New types: semigroups of matrices over semirings:
  - the integers: $\mathbb{Z}$
  - the Boolean semiring: the set $\{0, 1\}$ with:

$+$	0	1
0	0	1
1	1	1

$\times$	0	1
0	0	0
1	0	1

  - the max-plus semiring: $\mathbb{Z} \cup \{-\infty\}$ with operations max and $+$.
  - the min-plus semiring: $\mathbb{Z} \cup \{\infty\}$ with operations min and $+$.
  - the tropical max-plus semirings: $\{-\infty, 0, 1, \ldots, t\}$ with operations max and $+$ (truncated at t)
  - the tropical min-plus semirings: $\{0, 1, \ldots, t, \infty\}$ with operations min and $+$ (truncated at t)
  - $\mathbb{N}_{t,p} = \{0, 1, \ldots, t, t+1, \ldots, t+p-1\}$ for some t and p under $+$ and $\times$ mod $t = t + p$
- Partitioned binary relations (PBRs) as defined by Martin and Mazorchuk, 2011.

The Semigroups package for GAP - contributors

Semigroups has around 20 contributors:

- Manuel Delgado [Visualisation]
- James East [Partition semigroups]
- Attila Egri-Nagy [Partition semigroups, bug hunter]
- Julius Jonušas [Free inverse semigroups, free bands, ideals]
- Markus Pfeiffer [Partition and matrix semigroups]
- Ben Steinberg [Character tables for inverse semigroups]
- Michael Torpey [Congruences]
- Wilf Wilson [Small degree representations, maximal subsemigroups]

It is open-source and developed on bitbucket:

`http://bitbucket.org/james-d-mitchell/semigroups`

`http://tinyurl.com/semigroups`

Application 1: Endomorphisms of graphs, 1/1

A transformation f is **non-uniform** if there exist i, j such that $|(i)f^{-1}| \neq |(j)f^{-1}|$.

Conjecture (Araújo, Bentz, Cameron, Royle, Schaefer)

If G is a primitive group and f is any non-uniform transformation, then $\langle\, G, f \,\rangle$ contains a constant transformation.

The conjecture is false.

Application 2: Semigroups from digraphs, 1/4

Take a digraph D:

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4$$

For every edge $i \longrightarrow j$ get one transformation mapping i to j and fixing everything else:

$$D = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 4 \end{pmatrix} \right\}.$$

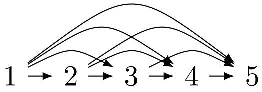
Define $S = \langle\, D \,\rangle$. We say that S is **arc-generated**.

What (if any) is the relationship between properties of the digraph D and the semigroup S?

Application 2: Examples, 2/4

Theorem (Howie '78, Solomon '96)

- *$T_n \setminus S_n$ is arc-generated by D if and only if D contains a strongly connected tournament;*
- *the* **semigroup of order-preserving transformations** $O_n := \{f \in T_n \setminus S_n : u \leq v \text{ if } (u)f \leq (v)f\}$ *is arc-generated by an undirected path of length n*
- *the* **Catalan semigroup** $C_n := \{f \in T_n \setminus S_n : v \leq (v)f, u \leq v \text{ if } (u)f \leq (v)f\}$ *is arc-generated by a directed path of length n*
- *the* **semigroup of non-decreasing transformations** $OI_n := \{f \in T_n \setminus S_n : v \leq (v)f\}$ *is arc-generated by the transitive tournament shown below.*



Application 2: The program, 3/4

Lemma

If D is a digraph with connected components $D_1, D_2, \ldots, D_n$, then $\langle\, D \,\rangle \cong \langle\, D_1 \,\rangle \times \langle\, D_2 \,\rangle \times \cdots \times \langle\, D_n \,\rangle$.

For $n \in \mathbb{N}$ and a property P of semigroups:

- get the set $\mathcal{D}_n$ of digraphs with n vertices up to isomorphism
- create all arc-generated semigroups $\mathcal{S}_n$
- find all the semigroups in $\mathcal{S}_n$ with property P
- look at all the digraphs which generate the semigroups in $\mathcal{S}_n$.

Application 2: Some results 4/4

Joint work with P. J. Cameron, A. Castillo-Ramirez, and M. Gadouleau

Theorem

Let D be a connected digraph. Then the following are equivalent:

(i) $\langle D \rangle$ *is inverse;*

(ii) $\langle D \rangle$ *is isomorphic to the semilattice of idempotents of the symmetric inverse monoid on a set of size* $n-1$ *(without the identity);*

(iii) $\langle D \rangle$ *is commutative;*

(iv) D *is a fan.*

In particular, note that $|\langle D \rangle| = 2^{n-1} - 1$.

Theorem

Let D be a digraph. Then D is acyclic if and only if $\langle D \rangle$ is $\mathscr{R}$-trivial.

Application 3: Hall matrices, 1/6

The Marriage Problem

Given two groups A and B of people:

- for every person a in A there is a subset $B_a \subseteq B$ such that a would be happy to marry anyone in B_a; and
- every person in B is happy to marry anyone who wants to marry them.

Is it possible to marry everyone so that everyone is happy?

Theorem (Hall '35)

Everyone is happy if and only if for any subset W of $\{ B_a : a \in A \}$,

$$|W| \leq \left| \bigcup_{B_a \in W} B_a \right|.$$

Application 3: Hall matrices, 2/6

For example, if $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$, and

$$B_1 = \{1, 2, 3\}, \quad B_2 = \{1, 4, 5\}, \quad B_3 = \{3, 5\},$$

then possible "marriages" are

$$(1, 1), (2, 4), (3, 5) \quad \text{and} \quad (1, 2), (2, 1), (3, 3).$$

But if

$$B_1 = \{2, 3, 4, 5\}, \quad B_2 = \{4, 5\}, \quad B_3 = \{5\}, \quad B_4 = \{4\},$$

then there are no possible "marriages" since if $W = \{B_2, B_3, B_4\}$, then

$$|W| = 3 > \left| \bigcup_{i=2}^{4} B_i \right| = |\{4, 5\}| = 2.$$

Application 3: Hall matrices 3/6

Suppose that

$$B_1 = \{1, 2, 3\}, \quad B_2 = \{2, 4\}, \quad B_3 = \{1, 2, 4\}, \quad B_4 = \{2, 4\}.$$

Then this can be described by a Boolean matrix where row i is the "characteristic" function of B_i:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

and "marriages" become permutation matrices contained this matrix.

We say that such a matrix is a **Hall matrix** if it contains a permutation matrix, and denote by $\mathbf{H}_n$ the monoid of all Hall matrices.

How many $n \times n$ Hall matrices are there?

Application 3: Hall matrices, 4/6

Lemma

If $\mathbf{R}_n$ *denotes the reflexive Boolean matrices and* S_n *the group of permutation matrices, then*

$$\mathbf{H}_n = \langle\, S_n, \mathbf{R}_n \,\rangle.$$

Proof.

- Up to rearranging the rows and columns, Hall matrices are just reflexive Boolean matrices;
- Rearranging the rows and columns, is just post- and premultiplying by a permutation matrix. $\square$

Application 3: Hall matrices, 5/6

Hall matrices

define a bijection $\Psi : \{1, \ldots, 2^{n^2-n}\} \longrightarrow \mathbf{R}_n$
$S := \varnothing$, $X := \varnothing$
for $i \in \{1, \ldots, 2^{n^2-n}\}$ **do**
    if $\Psi(i) \notin S$ **then**
        $X \leftarrow X \cup \{\Psi(i)\}$
        $S \leftarrow \langle\, X \,\rangle$
$X \leftarrow X$ up to rearranging rows and columns
return $\langle\, X, S_n \,\rangle$

It is well-known that $\mathbf{R}_n$ is $\mathscr{J}$-trivial, and so the Froidure-Pin Algorithm is the “best” algorithm for computing it!

Application 3: Hall matrices, 6/6

n	$d(\mathbf{R}_n) \leq$	$d(\mathbf{H}_n) \leq$	$\lvert\mathbf{H}_n\rvert$
2	2	2	7
3	8	4	247
4	38	6	37 823
5	1414	12	23 191 071
6	?	?	54 812 742 655

`https://oeis.org/A227414`

Thanks for listening!